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LETTERS TO THE EDITOR

ON THE NON-LINEAR DYNAMIC MODELLING OF THE FLEXIBLE CONNECTING ROD OF A SLIDER-CRANK MECHANISM WITH INPUT TORQUE

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1. INTRODUCTION

Mechanisms have been traditionally designed on the basis of the assumption that all members in the mechanism are rigid bodies. However, when simulating a mechanism that is running at high speed, a perturbative motion can be observed when rigid-body assumption is used. There will be some problems in the mechanism when the amplitudes of vibration are greater than the allowable limit. To obtain a more accurate prediction of the motion of a slider-crank mechanism, a dynamic analysis of the elastic connecting rod is necessary.

The dynamic analysis of a slider-crank mechanism has been studied extensively over the past 30 years, with much of the research going beyond the current paper to include a totally flexible mechanism. A survey can be found in the series of review articles provided by Erdman and Sander [1], Lowen and Jandrassits [2], Lowen and Chassapis [3], Thompson and Sung [4], and Erdman [5].

A constant angular velocity of the crank was assumed in all the above references, and both ends of the connecting rod were assumed to be simply supported, i.e., the moment and displacement were assumed to vanish at both ends. However, the realistic operating condition is that the crank is driven by an input torque, and one end of the connecting rod moves reciprocally with the slider along the horizontal guide.

The new aspect of the present paper is that the right-hand end of the rod, which is point B in Figure 1, is pinned with the slider and moves along the X-axis. The geometric constraint condition, describing the end point B moving along the X-axis, is introduced into Hamilton's principle to formulate the governing equations of the four separate models of the connecting rod. The rigid-body motion and flexible vibration are coupled in these formulations.

2. FORMULATION OF THE GOVERNING EQUATIONS

Four separate models for the in-plane motion of the slider-crank mechanism are used. They are the Timoshenko beam model, Euler beam model, simple-flexure model and rigid-body model. Hamilton's principle is employed to generate the governing equations of motion. The slider-crank mechanism is shown in Figure 1(a), and it consists of the rigid crank OA of length r; the flexible prismatic rod AB of length l; and the piston of mass

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M_4. Other symbols in this figure are as follows: N, normal force perpendicular to its direction of motion; F, external force acting on the piston; θ, crank angle; ϕ, angle between the X-axis and the undeformed axis of the connecting rod, and the external torque. A list of nomenclature is given in the Appendix.

2.1. *Geometric constraint*

The displacement field of the deformed beam is shown in Figure 1(b). $[\mathbf{i}, \mathbf{j}]$ are the unit vectors of the fixed co-ordinate system (OXY), and $[\mathbf{e}_r, \mathbf{e}_\theta]$ and $[\mathbf{e}_i, \mathbf{e}_j]$ are the unit vectors of the moving co-ordinates with origins at O and A respectively.

The displacement field of the Timoshenko beam is

$$u_1(x, y, t) = u(x, t) - y\psi(x, t), \qquad u_2(x, y, t) = v(x, t); \tag{1a, b}$$

where u and v represent the axial and transverse displacements of any point on the connecting rod, and ψ is the slope of the deflection curve due to bending deformation alone. The position vector of one arbitrary point P on the connecting rod is

$$\begin{aligned}\mathbf{R}(x, y, t) &= r\mathbf{e}_r + (x + u_1)\mathbf{e}_i + (y + u_2)\mathbf{e}_j \\ &= [r \cos\theta + (x + y)\cos\phi + (y + v)\sin\phi]\mathbf{i} + [r \sin\theta - (x + u)\sin\phi \\ &\quad + (y + v)\cos\phi]\mathbf{j}.\end{aligned}$$

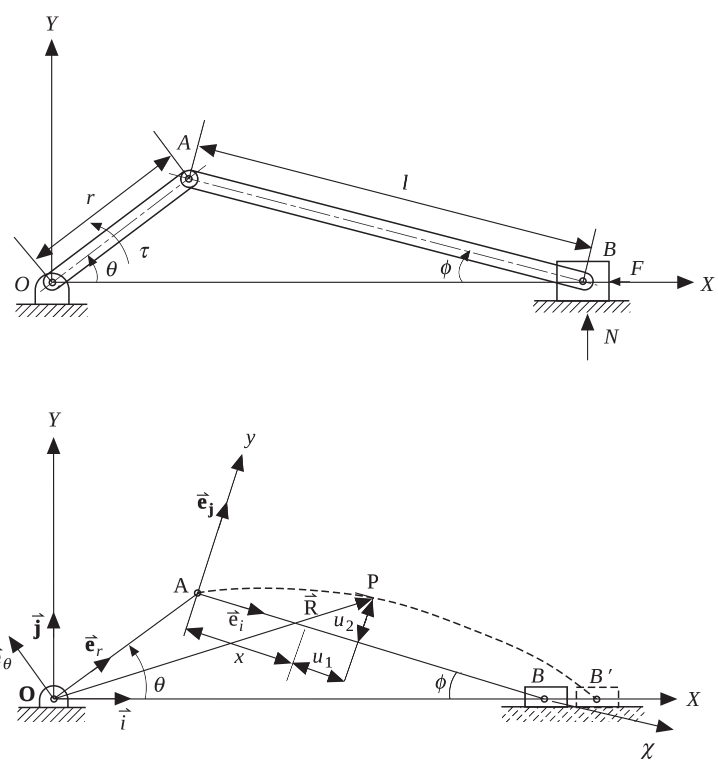

Figure 1. Slider-crank mechanism with a flexible connecting rod; (a) undeformed configuration; (b) deformed configuration.

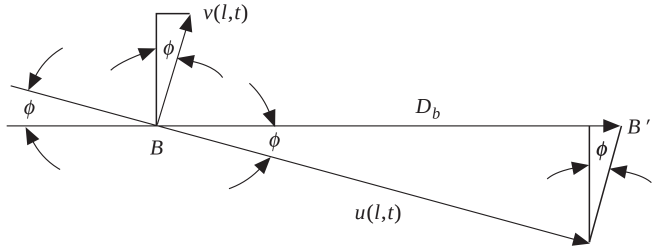

Figure 2. At end point B, the displacement relationship between D_b, $u(l, t)$ and $v(l, t)$.

Since the slider moves along the X-axis, point B, where the connecting rod is pinned to the slider, also moves along the X-axis. Thus, the constraint condition of point B is

$$0 = \mathbf{R}(l, 0, t) \cdot \mathbf{j} = r \sin\theta - [l + u(l, t)] \sin\phi + v(l, t)\cos\phi,$$

which means the displacement is always zero in the Y direction. Substituting the geometric relation

$$r \sin\theta = l \sin\phi \tag{2}$$

into the above equation, one has

$$v(l, t) = u(l, t) \tan\phi. \tag{3}$$

The above displacement relationship at point B can also be obtained from the geometric plot shown in Figure 2.

It is seen that $u(l, t) = v(l, t) = 0$ for the assumption of a simply supported end (Chu and Pan [6]; Tadjbakhsh and Younis [7]) and this possibility is also included in equation (3). If the axial displacement $u(x, t)$ is negligible (i.e. only the transverse displacement $v(x, t)$ is considered), the constraint condition at point B becomes $v(l, t) = 0$, and it becomes a simply supported end. In the present work, the axial and transverse displacements are considered simultaneously, so $u(l, t)$ and $v(l, t)$ are not independent and are related by equation (3). Taking the variation of equation (3), one has

$$\delta v(l, t) = \delta u(l, t) \tan\phi + u(l, t) \sec^2\phi\,\delta\phi. \tag{4}$$

Substituting (4) into $\delta\mathbf{R}(x, y, t)$, and taking $x = l$, one has

$$\delta\mathbf{R}(l, 0, t) = \delta u(l, t)\mathbf{e}_i + \delta v(l, t)\mathbf{e}_j = \delta u(l, t) \sec\phi\,\mathbf{i}.$$

Differentiating $\mathbf{R}(x, y, t)$ with respect to time, one gets the absolute velocity of the arbitrary point P on the connecting rod as

$$\begin{aligned}\mathbf{R}_t(x, y, t) &= r\theta_t\mathbf{e}_\theta + (u_t - y\psi_t)\mathbf{e}_i + v_t\mathbf{e}_j - \phi_t\mathbf{e}_k \times [(x + u - y\psi)\mathbf{e}_i + (y + v)\mathbf{e}_j] \\ &= [-r\theta_t \sin(\theta + \phi) + u_t - y\psi_t + \phi_t(y + v)]\mathbf{e}_i + [r\theta_t \cos(\theta + \phi) \\ &\quad + v_t - \phi_t(x + u - y\psi)]\mathbf{e}_j.\end{aligned} \tag{5}$$

Because the slider moves in the X direction and the component of the acceleration of the connecting rod in the Y direction is zero, the acceleration of point B can be written as

$$\mathbf{R}_{tt}(l, 0, t) = a_x(l, t) \sec\phi\,\mathbf{i},$$

where

$$a_x(l, t) = \mathbf{R}_{tt}(l, 0, t) \cdot \mathrm{e_i} = -r\theta_{tt} \sin(\theta + \phi) - r\theta_t^2 \cos(\theta + \phi)$$
$$+ u_{tt} + 2v_t\phi_t + v\phi_{tt} - (l + u)\phi_t^2. \tag{6}$$

The kinetic energy of the connecting rod can be expressed as

$$T_3 = \frac{1}{2}\int_V \rho \mathbf{R}_t(x, y, t) \cdot \mathbf{R}_t(x, y, t)\,\mathrm{d}V = \int_0^l T^*\,\mathrm{d}x, \tag{7}$$

where

$$T^* = (\rho A/2)\{[-r\theta_t \sin(\theta + \phi) + u_t + v\phi_t]^2 + [r\theta_t \cos(\theta + \phi) + v_t - \phi_t(x + u)]^2\}$$
$$+ (\rho I/2)[(\phi_t - \psi_t)^2 + \phi_t^2\psi^2]. \tag{8}$$

The Lagrangian strains are

$$\epsilon_{xx} = u_x - y\psi_x + \tfrac{1}{2}v_x^2, \qquad \epsilon_{yy} = 0, \qquad \epsilon_{xy} \approx \tfrac{1}{2}(v_x - \psi), \tag{9a–c}$$

where the higher order terms $u_x\psi$, $y\psi\psi_x$ are neglected in ϵ_{xy}. The strain energy of the connecting rod can be expressed as

$$U_3 = \frac{1}{2}\int_V \sigma_{ij}\epsilon_{ij}\,\mathrm{d}V = \int_0^l U^*\,\mathrm{d}x, \tag{10}$$

where

$$U^* = \tfrac{1}{2}[EA(u_x + \tfrac{1}{2}v_x^2)^2 + KGA(v_x - \psi)^2 + EI\psi_x^2]. \tag{11}$$

The kinetic energy of the crank with mass M_2 and mass momentum of inertia J_{2c} is

$$T_2 = \tfrac{1}{8}M_2 r^2\theta_t^2 + \tfrac{1}{2}J_{2c}\theta_t^2. \tag{12}$$

The kinetic energy of the slider is

$$T_4 = \tfrac{1}{2}M_4\mathbf{R}_t(l, 0, t) \cdot \mathbf{R}_t(l, 0, t) \tag{13}$$

and the virtual work done by the driving force F, and the friction force μN acting on the slider, and the external torque τ applied on the crank is

$$\delta W = [(F - \mu N)\mathbf{i} + N\mathbf{j}]\delta\mathbf{R}(l, 0, t) + \tau\delta\theta. \tag{14}$$

2.2. *Hamilton's principle*

By using Hamilton's principle, one can write

$$0 = \int_{t_1}^{t_2}\left\{\int_0^l \delta L\,\mathrm{d}s + \delta T_2 + \delta T_4 + \delta W\right\}\mathrm{d}t, \tag{15}$$

where $L(\theta, \theta_t, \phi, \phi_t, u, u_t, u_x, v, v_t, v_x, \psi, \psi_t, \psi_x) = T^* - U^*$ is the Lagrangian density of the slider-crank mechanism.

The point $(x, y) = (0, 0)$ is the common revoluting joint point of the rigid crank and the flexible connecting rod, and the values $u(0, t) = 0$ and $v(0, t)$ are specified, thus one has $\delta u(0, t) = 0$, and $\delta v(0, t) = 0$. The slope angles ψ at the end points $x = 0, l$ of the connecting rod are free and therefore $\delta\psi(0, t) \neq 0$, and $\delta\psi(l, t) \neq 0$.

Substituting equations (7, 10, 13, 14) into equation (15), using the constraint condition (4) at $x = l$ and introducing the damping terms C_x, C_y and C_ψ, which are proportional to the relative velocities u_t, v_t and ψ_t, one obtains the governing equations of the system

$$\int_0^l -\rho Ar\{r\theta_{tt} + [u_{tt} - v\phi_{tt} + 2v_t\phi_t + (x+u)\phi_t^2]\sin(\theta+\phi)$$

$$+ [u_{tt} - \phi_{tt}(x+u) - \phi_t^2 u]\cos(\theta+\phi)\}\,\mathrm{d}x + M_4 r\{r\theta_{tt} + [u_{tt} - v\phi_{tt}$$

$$+ 2u_t\phi_t + (x+u)\phi_t^2]\sin(\theta+\phi) + [u_{tt} - \phi_{tt}(x+u) - \phi_t^2 u]\cos(\theta+\phi)\}$$

$$- ((M_2/4)r^2 + J_{2c})\theta_{tt} + \tau = 0, \qquad (16a)$$

$$\int_0^l - \rho A[-r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta+\phi) + u_{tt} + 2v_t\phi_t + v\phi_{tt}]v + 2u_t v_t$$

$$+ 2r\theta_t u_t\cos(\theta+\phi) - (x+u)[r\theta_{tt}\cos(\theta+\phi) - r\theta_t^2\sin(\theta+\phi) + v_{tt}$$

$$- (x+u)\phi_{tt}] + \rho I(\phi_{tt} - \psi_{tt} + \phi_{tt}\psi^2 + 2\phi_t\psi_t\psi)\}\,\mathrm{d}x$$

$$- M_4\{[-r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta+\phi) + u_{tt} + 2v_t\phi_t + v\phi_{tt}]v$$

$$+ 2u_t v_t + 2r\theta_t u_t\cos(\theta+\phi) - (x+u)[r\theta_{tt}\cos(\theta+\phi) - r\theta_t^2\sin(\theta+\phi)$$

$$+ v_{tt} - (x+u)\phi_{tt}]\} + \{EAv_x(v_x + \tfrac{1}{2}v_x^2) + KGA(v_x - \psi) + M_4[-r\theta_{tt}\cos(\theta+\phi)$$

$$+ r\theta_t^2\sin(\theta+\phi) + 2\phi_t u_t + v\phi_t^2 - v_{tt} + (x+u)\phi_{tt}]\}u(l,t)\sec^2\phi = 0, \qquad (16b)$$

$$\rho A[r\theta_t^2\cos(\theta+\phi) - 2\phi_t v_t + (x+u)\phi_t^2 + r\theta_{tt}\sin(\theta+\phi) - u_{tt} - v\phi_{tt}]$$

$$- C_x u_t + EA(\partial/\partial x)(u_x + \tfrac{1}{2}v_x^2) = 0, \qquad (16c)$$

$$\rho A[-r\theta_{tt}\cos(\theta+\phi) + r\theta_t^2\sin(\theta+\phi) + 2\phi_t u_t + v\phi_t^2 - v_{tt} + (x+u)\phi_{tt}]$$

$$- C_y v_t + EA[v_x(\partial/\partial x)(u_x + \tfrac{1}{2}v_x^2) + v_{xx}(u_x + \tfrac{1}{2}v_x^2)] + KGA(v_{xx} - \psi_x) = 0,$$

$$\rho I(\phi_t^2\psi + \phi_{tt} - \psi_{tt}) - C_\psi\psi_t - KGA(\psi - v_x) + (\partial/\partial x)(EI\psi_x) = 0; \qquad (16e)$$

and the boundary conditions

$$u(0,t) = 0, \qquad v(0,t) = 0, \qquad \psi_x(0,t) = 0, \qquad \psi_x(l,t) = 0, \qquad (17a\text{–}d)$$

$$[(F - \mu N)\sec\phi - M_4 a_x(l,t)\sec^2\phi] - KGA[v_x(l,t) - \psi(l,t)]\tan\phi$$

$$- EA[u_x(l,t) + \tfrac{1}{2}v_x^2(l,t)](1 + v_x(l,t)\tan\phi) = 0. \qquad (17e)$$

The non-linear partial differential equations (16a–e) include the second-order spatial derivatives of all the variables u, v and ψ. The five boundary conditions (17a–e) and one constraint condition (3) are satisfied to solve those three equations. Equations (16a, b) describe the rigid-body motions of the crank and connecting rod respectively, while equations (16c–e) describe the flexural vibration of the connecting rod which is modelled by Timoshenko beam theory. It is seen that the rigid-body motion and flexural vibration are coupled. Even though the input torque is absent, the coupling still exists. Boundary condition (17d) states that there is zero moment at revoluting joint B, while equation (17e) describes the dynamic behavior of the slider in the X direction.

2.3. *Euler beam theory*

If the slenderness of the beam is very small, the Euler beam theory can be used to describe bending of the connecting rod by setting $\psi = v_x$ and neglecting the rotating inertia effect of $\rho I(\phi_t^2\psi + \phi_{tt} - \psi_{tt})$, one then obtains the following governing equations of θ, ϕ, u and v respectively,

$$\int_0^l -\rho Ar\{r\theta_{tt} + [u_{tt} - v\phi_{tt} + 2v_t\phi_t + (x+u)\phi_t^2]\sin(\theta+\phi) + [u_{tt} - \phi_{tt}(x+u) - \phi_t^2 u]\cos(\theta+\phi)\}\,\mathrm{d}x + M_4 r\{r\theta_{tt} + [u_{tt} - v\phi_{tt} + 2u_t\phi_t + (x+u)\phi_t^2]\sin(\theta+\phi) + [u_{tt} - \phi_{tt}(x+u) - \phi_t^2 u]\cos(\theta+\phi)\} - (M_2/4)r^2 + J_{2c})\theta_{tt} + \tau = 0, \tag{18a}$$

$$\int_0^l -\rho A\{[-r\theta_{tt}\sin(\theta+\phi) + 2v_t\phi_t - r\theta_t^2\cos(\theta+\phi) + u_{tt} + v\phi_{tt}]v + 2u_t v_t + 2r\theta_t u_t\cos(\theta+\phi) - (x+u)[r\theta_{tt}\cos(\theta+\phi) - r\theta_t^2\sin(\theta+\phi) + v_{tt} - (x+u)\phi_{tt}] + \rho I[\phi_{tt} - v_{xtt} + \phi_{tt}v_x^2 + 2\phi_t v_{xt} v_x]\}\,\mathrm{d}x - M_4\{[-r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta+\phi) + u_{tt} + 2v_t\phi_t + v\phi_{tt}]v + 2u_t v_t + 2r\theta_t u_t\cos(\theta+\phi) - (x+u) \times [r\theta_{tt}\cos(\theta+\phi) - r\theta_t^2\sin(\theta+\phi) + v_{tt} - (x+u)\phi_{tt}]\} + \{EAv_x(v_x + \tfrac{1}{2}v_x^2) + KGA(v_x - \psi) + M_4[-r\theta_{tt}\cos(\theta+\phi) + r\theta_t^2\sin(\theta+\phi) + 2\phi_1 u_t + v\phi_t^2 - v_{tt} + (x+u)\phi_{tt}]\}u(l,t)\sec^2\phi = 0, \tag{18b}$$

$$\rho A[r\theta_t^2\cos(\theta+\phi) - 2\phi_t v_t + (x+u)\phi_t^2 + r\theta_{tt}\sin(\theta+\phi) - u_{tt} - v\phi_{tt}] - C_x u_t + EA[(\partial/\partial x)(u_x + \tfrac{1}{2}v_x^2)] = 0, \tag{18c}$$

$$\rho A[r\theta_t^2\sin(\theta+\phi) - r\theta_{tt}\cos(\theta+\phi) + 2\phi_t u_t + v\phi_t^2 - v_{tt} + (x+u)\phi_{tt}] - C_y v_t + EA[v_x(\partial/\partial x)(u_x + \tfrac{1}{2}v_x^2) + v_{xx}(u_x + \tfrac{1}{2}v_x^2)] - EIv_{xxxx} = 0; \tag{18d}$$

and the boundary conditions at $x = 0, l$, respectively,

$$u(0,t) = 0, \qquad v(0,t) = 0, \qquad v_{xx}(0,t) = 0, \qquad v_{xx}(l,t) = 0, \tag{19a–d}$$

$$[(F - \mu N)\sec\phi - M_4 a_x(l,t)\sec^2\phi] - EA[u_x(l,t) + \tfrac{1}{2}v_x^2(l,t)](1 + v_x(l,t)\tan\phi) = 0. \tag{19e}$$

The same differential equations (18c, d) of longitudinal and transverse vibrations were derived by Chu and Pan [6], by using the equilibrium of forces and moments acting on the differential element at x. Also, the rigid-body motion and flexural vibration are coupled.

2.4. *Simple flexure model*

For simplicity, neglecting the axial displacement $u(x, t)$ and the rotating inertia effect, one can obtain the governing equations of θ, ϕ and v respectively, and the boundary conditions as

$$\int_0^l -\rho Ar\{(r\theta_{tt} - v\phi_{tt} + 2v_t\phi_t + x\phi_t^2)\sin(\theta+\phi) - x\phi_{tt}\cos(\theta+\phi)\}\,\mathrm{d}x$$

$$+ M_4 r\{r\theta_{tt} + [-v\phi_{tt} + x\phi_t^2]\sin(\theta+\phi) - x\phi_{tt}\cos(\theta+\phi)\}$$

$$- ((M_2/4)r^2 + J_{2c})\theta_{tt} + \tau = 0, \tag{20a}$$

$$\int_0^l -\rho A\{[-r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta+\phi) + 2v_t\phi_t + v\phi_{tt}]v - x[r\theta_{tt}\cos(\theta+\phi)$$

$$- r\theta_t^2\sin(\theta+\phi) + v_{tt} - x\phi_{tt}]\} + \rho I[\phi_{tt} - \psi_{tt} + \phi_{tt}\psi^2 + 2\phi_t\psi_t\psi\}\,\mathrm{d}x$$

$$- M_4\{[-r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta+\phi) + 2v_t\phi_t + v\phi_{tt}]v$$

$$- x[r\theta_{tt}\cos(\theta+\phi) - r\theta_t^2\sin(\theta+\phi) + v_{tt} - x\phi_{tt}]\} = 0, \tag{20b}$$

$$\rho A[r\theta_t^2\sin(\theta+\phi) - r\theta_{tt}\cos(\theta+\phi) + v\phi_t^2 - v_{tt} + x\phi_{tt}]$$

$$- C_y v_t + \tfrac{3}{2}EAv_x^2 v_{xx} - EIv_{xxxx} = 0, \tag{20c}$$

$$v(0, t) = 0, \qquad v_{xx}(0, t) = 0, \qquad v(l, t) = 0, \qquad v_{xx}(l, t) = 0, \tag{21a–d}$$

$$[(F - \mu N)\sec\phi - M_4\bar{a}_x(l, t)\sec^2\phi] - \tfrac{1}{2}EAv_x^2(l, t)(1 + v_x(l, t)\tan\phi) = 0, \tag{22}$$

where

$$\bar{a}_x(l, t) = -r\theta_{tt}\sin(\theta+\phi) - r\theta_t^2\cos(\theta + 2\phi) + 2v_t\phi_t + v\phi_{tt} - l\phi_t^2.$$

Since all the effects of longitudinal displacement $u(x, t)$ are neglected, the geometric constraint (3) is reduced to the trivial transverse displacement condition (21c), which is in the nature of a simply supported condition. With four boundary conditions (21a–d), the fourth-order equation (20c) can be solved.

The additional equation (22) is the dynamic equilibrium equation for the slider, which includes the elastic force, the normal force N, the external loading F, and the inertia force of the slider. In order to bring the effect of equation (22) into governing equation (20c), one lets

$$p(x, t) = \tfrac{1}{2}EAv_x^2(x, t), \tag{23}$$

and rewrites equation (20) as

$$\rho A[r\theta_t^2\sin(\theta+\phi) - r\theta_{tt}\cos(\theta+\phi) + v\phi_t^2 - v_{tt} + x\phi_{tt}]$$

$$- C_y v_t + 3p(x, t)v_{xx} - EIv_{xxxx} = 0. \tag{24}$$

The value of $p(x, t)$ in equation (24), instead of boundary condition (22) at $x = l$, can be obtained by the integration of $(\partial/\partial x)p(x, t)$ from x to l. Then, utilizing the value of $p(l, t)$ from equation (23) by taking $x = l$, and the expression of equation (22), one has

$$p(x, t) = p(l, t) - \int_x^l \frac{\partial}{\partial x} p(x, t)\,\mathrm{d}x$$

$$= \frac{1}{1 + v_x(l, t)\tan\phi}\,[(F - \mu N)\sec\phi - M_4 \bar{a}_x(l, t)\sec^2\phi] - \int_x^l EAv_x v_{xx}\,\mathrm{d}x.$$

Badlani and Midha [8] studied the dynamic behavior of a slider-crank mechanism with an initially curved connecting rod. Only the transverse deflection was considered, and this was measured from the initially curved axis of the unstressed connecting rod. Hsieh and Shaw [9] studied the dynamic stability and non-linear resonance of equation (24) with the assumption that the crank was operated at constant angular velocity and the connecting rod was made of a viscoelastic Kelvin-Voigt material.

2.5. *Rigid-body model*

When neglecting all flexibility, one obtains the rigid body motion of the slider-crank mechanism. Assuming the crank and the connecting rod have uniform cross-sectional area, and integrating equations (20a, b), one obtains the governing equations for θ and ϕ, respectively, as

$$M_3 r\{r\theta_{tt} + (l/2)\phi_t^2 \sin(\theta + \phi) - (l/2)(\phi_{tt}\cos(\theta + \phi)\} + M_4 r\{r\theta_{tt} + l\phi_t^2 \sin(\theta + \phi)$$
$$- l\phi_{tt}\cos(\theta + \phi)\} - ((M_2/3)r^2)\theta_{tt} + \tau = 0, \tag{25a}$$
$$M_3\{(rl/2)\theta_{tt}\cos(\theta + \phi) - (rl/2)\theta_t^2 \sin(\theta + \phi) - (l^2/3)\phi_{tt}\} + M_4\{rl\theta_{tt}\cos(\theta + \phi)$$
$$- rl\theta_t^2 \sin(\theta + \phi) - l^2\phi_{tt}\} = 0 \tag{25b}$$

where $M_3 = \rho Al$ is the mass of the uniform connecting rod. Also, equations (25a, b) can be obtained directly by using the Lagrange equation

$$\partial T_r/\partial\theta - (\mathrm{d}/\mathrm{d}t)(\partial T_r/\partial\theta_t) = \tau, \qquad \partial T_r/\partial\phi - (\mathrm{d}/\mathrm{d}t)(\partial T_r/\partial\phi_t) = 0,$$

where

$$T_r = (M_1 r^2/6)\theta_t^2 + (M_2 l^2/24)\phi_t^2 + (M_2/2)(r^2\sin^2(\theta)\theta_t + (l^2/4)\phi_t^2$$
$$+ rl\sin(\theta)\sin(\phi)\theta_t\phi_t) + \tfrac{1}{2}M_3(r^2\sin^2(\theta)\theta_t^2 + l^2\sin^2(\phi)\phi_t^2$$
$$+ 2rl\sin(\theta)\sin(\phi)\theta_t\phi_t)$$

is the kinetic energy of the rigid body motion of the slider-crank mechanism. The constraint equation (2) and governing equations (25a, b) for a constrained mechanical system are a mixed set of algebraic and differential equations. Several numerical methods have been used to solve such a system of equations (Wehage and Haug, [10]; Orlandea *et al*. [11]). An in-depth discussion of these methods is not within the scope of this paper. The main purpose here is to formulate the model of the slider-crank mechanism with external torque.

3. DISCUSSION

From the physical meaning, it is valid to have zero moment (16a) at the revoluting joint B. However, the boundary condition (16b) and the constraint condition (3) obtained in the present paper are different from those of Jasinski *et al*. [12] and Badlani and Kleinhenz [5], who assumed zero displacement, $v(l, t) = 0$, and utilized Newton's second law to balance the axial load, shear load and piston inertia force at $x = l$. Other investigators (Badlani and Midha [8], Zhu and Chen [13], Tadjbakhsh and Younis [7]) assumed that the axial displacement is small compared to the transverse displacement, and neglected the contribution of the axial displacement to the inertia forces. Therefore, the right end of the connecting rod was assumed to be simply supported.

When assuming zero transverse displacement at the end point B, $v(l, t) = 0$, one then has $u(l, t) = 0$ from equation (3). Since $u(l, t) = v(l, t) = 0$, there is no deformation at the end point B, and the piston position from the elastic assumption of the connecting rod will be the same as that from the rigid body assumption. However, in the present work the end point B of the connecting rod moves freely and the displacements are related by equation (3). Therefore, the piston position due to the elastic deformation could be predicted. With the help of Figure 2, it is convenient to define the horizontal displacement at point B as

$$D_b = u(l, t)/\cos\phi. \tag{26}$$

Using the displacement relationship shown in Figure 2, one has $v(l, t) = u(l, t)\tan\phi$, given earlier as equation (3). The transient transverse amplitude, $v(l, t)$, is then obtained from equation (26) and the transient horizontal displacement D_b at the end point B along the X-axis is obtained.

A slider-crank mechanism with the usual proportions (connecting rod longer than crank) has two limiting positions, both occurring when the crank and the connecting rod are collinear, i.e., $\phi = 0$. At the limiting positions, one has zero transverse displacement, $v(l, t) = 0$ from equation (3), and the horizontal displacement equals the axial deformation, $D_b = u(l, t)$ from equation (26).

4. CONCLUSIONS

This paper provides four dynamic models of the flexible connecting rod. The geometric constraint condition, describing the end point B as it moves along the X-axis, is introduced into Hamilton's principle to formulate the governing equations of the connecting rod. The conclusions of the present work are

(1) From the non-linear dynamic modelling, it is seen that the crank is driven by an input torque, and the energy is transferred to the connecting rod, so that the rigid-body motion and flexural vibration are coupled.

(2) A motion-induced-vibration problem will arise in the slider-crank mechanism. Even though the input torque is absent, the rigid-body motion and flexural vibration are also coupled.

(3) A new definition of a revised boundary condition at the joint between the connecting rod and slider is offered. The boundary condition of the connecting rod moving with the slider is a time-dependent boundary support, and not a pure simple support.

(4) Stated in other words, the boundary constraint at end point B is released and is instead $v(l, t) = u(l, t)\tan\phi$.

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REFERENCES

1. A. G. Erdman and G. N. Sandor 1972 *Mechanics and Machine Theory* **7**, 19–33. Kieneto-elastodynamics—a review of the state of the art trends.
2. G. G. Lowen and W. G. Jandrasits 1972 *Mechanics and Machine Theory* **7**, 3–17. Survey of investigation into the dynamic behavior of mechanism containing links with distributed mass and elasticity.
3. G. G. Lowen and C. Chassapis 1986 *Mechanics and Machine Theory* **21**, 33–42. The elastic behavior of linkages: an update.
4. B. S. Thompson and C. K. Sung 1986 *Mechanics and Machine Theory* **21**, 351–359. A survey of finite element techniques for mechanism design.
5. A. G. Erdman 1993 *Modern kinematics*, *developments in the last forty years*. Wiley Interscience, Chapter 9.
6. S. C. Chu and K. C. Pan 1975 *ASME Journal of Engineering for Industry*, 542–550. Dynamic response of a high-speed slider-crank mechanism with an elastic connecting rod.
7. I. G. Tadjbakhsh and C. J. Younis 1986 *ASME Journal of Mechanics Transaction and Automatic in Design*, 487–496. Dynamic stability of the flexible connecting rod of a slider-crank mechanism.
8. M. Badlani and A. Midha 1982 *ASME Journal of Mechanics Design*, 159–167. Member initial curvature effects on the elastic slider-crank mechanism response.
9. S. R. Hsieh and S. W. Shaw 1994 Journal of Sound and Vibration **170**, 25–49. The dynamic stability and non-linear resonance of a flexible connecting rod: single-mode model.
10. R. A. Wehage and E. J. Haug 1982 *ASME Journal of Mechanics Design* **104**, 247–253. Generalized coordinate partitioning for dimension reduction in analysis of constrained dynamic system.
11. N. Orlandea, M. A. Clace and D. A. Colaham 1979 *ASME Journal of Engineering for Industry* **99**, 773–783. A sparsify-oriented approach to the dynamic analysis and design of mechanical system, Parts I and II.
12. P. W. Jasinski, H. C. Lee and G. N. Sandor 1971 *ASME Journal of Engineering for Industry*, 636–644. Vibrations of elastic connecting rod of a high-speed slider-crank mechanism.
13. Z. G. Zhu and Y. Chen 1983 *ASME Journal of Mechanics Transactions and Automatic in Design*, 637–640. The stability of the motion of a connecting rod.

APPENDIX: NOMENCLATURE

XOY	co-ordinate system
xAy	moving co-ordinate system
A	cross-sectional area of the connecting rod (m^2)
C_x, C_y, C_ψ	coefficients for viscous damping (Ns/m)
$\mathbf{e}_i$, $\mathbf{e}_j$	unit vectors in the x and y directions, respectively
$\mathbf{e}_r$, $\mathbf{e}_\theta$	unit vectors of the rotation co-ordinates originating at O
E	Young's modulus (N/m^2)
F	external force acting on the slider
G	shear modulus of elasticity (N/m^2)
$\mathbf{i}$, $\mathbf{j}$	unit vectors in the X and Y directions, respectively
I	area moment of inertia about neutral axis (m^4)
l	length of connecting rod (m)
M_2, M_4	mass of crank and slider, respectively
N	normal force acting on the slider
r	length of the crank (m)
$\mathbf{R}$	position vector related to XOY coordinate system
t	time (s)
u, v	longitudinal and transverse displacements of the rod, respectively
θ	crank angle
μ	the coefficient of sliding friction
ρ	mass density of connecting rod
ϕ	angle between X-axis and the undeformed axis of the connecting rod
ψ	the rotation of the cross-section due to bending